CHAPTER 1

Integrals

Recap Notes

INDEFINITE INTEGRAL

- Integration is the inverse process of differentiation.
  i.e., $\frac{d}{dx}F(x) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C$,
  where C is the constant of integration.
  Integrals are also known as antiderivatives.

Some Standard Integrals

- $\int dx = x + C$, where 'C' is the constant of integration
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log_e a} + C$, where $a > 0$
- $\int \frac{1}{x} dx = \log_e |x| + C$, where $x \neq 0$
- $\int \sin x\, dx = -\cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \text{cosec}^2 x\, dx = -\cot x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$, where $|x| < 1$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C$
- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C = -\text{cosec}^{-1} x + C$, where $|x| > 1$
- $\int \tan x dx = \log|\sec x| + C = -\log|\cos x| + C$
- $\int \cot x\, dx = \log|\sin x| + C$
- $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$
- $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

Properties of Indefinite Integral

(i) $\int f'(x)dx = f(x) + C$

(ii) $\int f(x)dx = \int g(x)dx + C$, f and g are indefinite integrals with the same derivative.

(iii) $\int [f(x) + g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$

(iv) $\int k \cdot f(x)\,dx = k\int f(x)\,dx$, k being any real number.

METHODS OF INTEGRATION

Integration by Substitution

- The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Integrals	Substitution
$\int f(ax+b)dx$	$ax + b = t$
$\int f(g(x))g'(x)dx$	$g(x) = t$
$\int \frac{f'(x)}{f(x)} dx$	$f(x) = t$
$\int (f(x))^n f'(x)dx$	$f(x) = t$
$\int (px+q)\sqrt{cx+d}\,dx$ or $\int \frac{px+q}{\sqrt{cx+d}} dx$	$px + q = A(cx + d) + B$. Find A and B by equating coefficients of like powers of x on both sides.

$\int \frac{1}{(px+q)\sqrt{cx+d}}dx$ or $\int \frac{1}{(px^2+qx+r)\sqrt{cx+d}}dx$	$cx + d = t^2$
$\int \frac{1}{(px+q)\sqrt{cx^2+dx+e}}dx$	$px+q=\frac{1}{t}$
$\int \frac{1}{(px^2+q)\sqrt{cx^2+d}}dx$	$x=\frac{1}{t}$ and then $c + dt^2 = u^2$
$\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ or $\int (px+q)\sqrt{ax^2+bx+c}\,dx$	$(px + q) = A\frac{d}{dx}(ax^2+bx+c)+B$

Integration using Trigonometric Identities

- When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can be easily integrated. Some of the identities useful for this purpose are given below :

  (i) $2\sin^2\left(\frac{x}{2}\right)=(1-\cos x)$

  (ii) $2\cos^2\left(\frac{x}{2}\right)=(1+\cos x)$

  (iii) $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

  (iv) $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$

  (v) $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

  (vi) $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

- **Some Special Substitutions**

Expression	Substitution
$\sqrt{a^2-x^2}$	$x = a \sin\theta$ or $a \cos\theta$
$\sqrt{a^2+x^2}$ or (a^2+x^2)	$x = a \tan\theta$ or $a \cot\theta$
$\sqrt{x^2-a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a \sin^2\theta$ or $a \cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a \tan^2\theta$ or $a \cot^2\theta$
$\sqrt{\frac{a-x}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$ or $\sqrt{(a-x)(x-b)}$	$x = a \cos^2\theta + b \sin^2\theta$

Integrals of Some Particular Functions

(i) $\int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(ii) $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

(iii) $\int \frac{1}{\sqrt{x^2-a^2}}dx=\log\left|x+\sqrt{x^2-a^2}\right|+C$

(iv) $\int \frac{1}{\sqrt{x^2+a^2}}dx=\log\left|x+\sqrt{x^2+a^2}\right|+C$

(v) $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(vi) $\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

Integration by Partial Fractions

- If $f(x)$ and $g(x)$ are two polynomials such that $\deg f(x) \geq \deg g(x)$, then we divide $f(x)$ by $g(x)$.

  $\therefore \frac{f(x)}{g(x)}=\text{Quotient}+\frac{\text{Remainder}}{g(x)}$

- If $f(x)$ and $g(x)$ are two polynomials such that the degree of $f(x)$ is less than the degree of $g(x)$, then we can evaluate $\int \frac{f(x)}{g(x)}dx$ by decomposing $\frac{f(x)}{g(x)}$ into partial fraction.

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a\neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ can not be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Integration by Parts

- If u and v are two differentiable functions of x, then

$$\int (uv)\, dx = \left[u \cdot \int v dx\right] - \int \left\{\frac{du}{dx} \cdot \int v dx\right\} dx \quad .$$

In order to choose 1st function, we take the letter which comes first in the word ILATE.
I – Inverse Trigonometric Function
L – Logarithmic Function
A – Algebraic Function
T – Trigonometric Function
E – Exponential Function

- **Integral of the type**

$$\int e^x (f(x) + f'(x))dx = e^x f(x) + C$$

INTEGRALS OF SOME MORE TYPES

(i) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

DEFINITE INTEGRAL

- Let $F(x)$ be integral of $f(x)$, then for any two values of the independent variable x, say a and b, the difference $F(b) - F(a)$ is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)dx$.

Here, $x = a$ is the lower limit and $x = b$ is the upper limit of the integral.

FUNDAMENTAL THEOREM OF CALCULUS

- **First Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

- **Second Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$

EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

- When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $t = f(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $f(a)$ and $f(b)$ respectively.

SOME PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

In Particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

(vi) $\int_{-a}^a f(x)dx = \begin{cases} 0 & , \text{ if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{ if } f(-x) = f(x) \end{cases}$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

(viii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$

Practice Time

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions (MCQs)

1. Evaluate :

$\int (3\sin x - 2\cos x + 4\sec^2 x - 5\operatorname{cosec}^2 x)\,dx$

(a) $-3\cos x - 2\sin x + 4\tan x + 5\cot x + C$

(b) $3\cos x + 2\sin x + 4\tan x + 5\cot x + C$

(c) $-3\cos x + 2\sin x - 4\tan x - 5\cot x + C$

(d) $-3\cos x - 2\sin x - 4\tan x - 5\cot x + C$

2. Evaluate : $\int (2^x + 2^{-x})^2\,dx$

(a) $\dfrac{1}{2\log 2}(2^{2x} - 2^{-2x}) + C$

(b) $\dfrac{1}{2\log 2}(2^{2x} - 2^{-2x}) + 2x + C$

(c) $\dfrac{1}{2\log 2}(2^{2x} + 2^{-2x}) + 2x + C$

(d) $\dfrac{1}{2\log 2}(2^{2x} + 2^{-2x}) + C$

3. Find the value of $\int \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\,dx$.

(a) $\tan x - \cot x + C$ (b) $-\tan x + \cot x + C$

(c) $\tan x + \cot x + C$ (d) $-\tan x - \cot x + C$

4. Evaluate : $\displaystyle\int_2^4 \frac{x}{x^2+1}\,dx$

(a) $\dfrac{1}{2}\log\left(\dfrac{17}{5}\right)$ (b) $\dfrac{1}{2}\log\left(\dfrac{5}{17}\right)$

(c) $\log\left(\dfrac{17}{5}\right)$ (d) $\log\left(\dfrac{5}{17}\right)$

5. $\int x e^{x^2}\,dx$ is equal to

(a) $-\dfrac{e^{x^2}}{2} + C$ (b) $\dfrac{e^{x^2}}{2} + C$

(c) $\dfrac{e^x}{2} + C$ (d) $-\dfrac{e^x}{2} + C$

6. Evaluate : $\displaystyle\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}\,dx$

(a) $\dfrac{2}{\cos\frac{x}{2} + \sin\frac{x}{2}} + C$ (b) $\dfrac{-2}{\cos\frac{x}{2} - \sin\frac{x}{2}} + C$

(c) $\dfrac{-2}{\cos\frac{x}{2} + \sin\frac{x}{2}} + C$ (d) $\dfrac{2}{\cos\frac{x}{2} - \sin\frac{x}{2}} + C$

7. Evaluate: $\displaystyle\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}\,dx$

(a) $57 - 5\sqrt{5}$ (b) $\dfrac{57 - \sqrt{5}}{5}$

(c) $\dfrac{57 + 5\sqrt{5}}{5}$ (d) $\dfrac{57 - 5\sqrt{5}}{5}$

8. Evaluate : $\int 2^{2^{2^x}} 2^{2^x} 2^x\,dx$

(a) $\dfrac{1}{(\log 2)^3} 2^{2^{2^x}} + C$ (b) $\dfrac{1}{(\log 2)^3} 2^{2^x} + C$

(c) $\dfrac{1}{(\log 2)^2} 2^{2^x} + C$ (d) $\dfrac{1}{(\log 2)^4} 2^{2^{2^x}} + C$

9. Evaluate : $\int 2^{(x+3)}\,dx$

(a) $\dfrac{2^x}{\log 2} + C$ (b) $\dfrac{2^3}{\log 2} + C$

(c) $\dfrac{2^{(x+3)}}{\log 2} + C$ (d) $\dfrac{2^{(x-3)}}{\log 2} + C$

10. Evaluate : $\int \sin^3 x \cos^3 x\,dx$

(a) $\dfrac{-1}{32}\left\{\dfrac{-3}{2}\cos 2x + \dfrac{1}{6}\cos 6x\right\} + C$

(b) $\dfrac{1}{32}\left\{\dfrac{-3}{2}\cos 6x + \dfrac{1}{6}\cos 2x\right\} + C$

(c) $\dfrac{1}{32}\left\{\dfrac{-3}{2}\cos 2x + \dfrac{1}{6}\cos 6x\right\} + C$

(d) None of these

11. Evaluate : $\int \sqrt{(x-3)(5-x)}\,dx$

(a) $\frac{1}{2}(x-4)\sqrt{(x-3)(5-x)}+\frac{1}{2}\cos^{-1}(x-4)+C$

(b) $\frac{1}{2}(x-4)\sqrt{(x-3)(5-x)}+\frac{1}{2}\sin^{-1}(x-4)+C$

(c) $\frac{1}{2}\sqrt{(x-3)(5-x)}+\frac{1}{2}\sin^{-1}(x-4)+C$

(d) None of these

12. Evaluate : $\int_0^{\pi/4} \tan^3 x\, dx$

(a) (1 – log2) (b) (1 + log2)

(c) $\frac{1}{2}(1-\log 2)$ (d) $\frac{1}{2}(1+\log 2)$

13. Evaluate : $\int \frac{\cot x}{\sqrt[3]{\sin x}}\, dx$

(a) $\frac{-3}{\sqrt[3]{\sin x}}+C$ (b) $\frac{-2}{\sin^3 x}+C$

(c) $\frac{3}{\sin^{1/3} x}+C$ (d) None of these

14. Evaluate : $\int x^2(ax+b)^{-2}dx$

(a) $\frac{1}{a^3}\left(ax+b-\frac{b^2}{ax+b}-2b\log(ax+b)\right)+C$

(b) $\frac{1}{a^3}\left(ax+b+\frac{b^2}{ax+b}-2b\log(ax+b)\right)+C$

(c) $\frac{1}{a^3}\left(ax+b+\frac{b^2}{ax+b}+2b\log(ax+b)\right)+C$

(d) $\frac{1}{a^3}\left(ax+b-\frac{b^2}{ax+b}+2b\log(ax+b)\right)+C$

15. Evaluate : $\int_0^1 \left\{e^x+\sin\frac{\pi x}{4}\right\}dx$

(a) $e+1+\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$ (b) $e-1-\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$

(c) $e+1-\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$ (d) $e-1+\frac{2\sqrt{2}}{\pi}-\frac{4}{\pi}$

16. Evaluate : $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

(a) $\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+c$ (b) $\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+c$

(c) $\frac{2}{3}\cos^{-1}\left(\frac{x}{a}\right)^{3/2}+c$ (d) $\frac{3}{2}\cos^{-1}\left(\frac{x}{a}\right)^{3/2}+c$

17. Evaluate : $\int_0^2 e^{3-4x}dx$

(a) $\frac{-1}{4}(e^5-e^3)$ (b) $\frac{1}{4}(e^5-e^3)$

(c) $\frac{1}{4}(e^{-5}-e^3)$ (d) $\frac{-1}{4}(e^{-5}-e^3)$

18. The value of $\int_0^{2\pi}\frac{dx}{e^{\sin x}+1}$ is

(a) π (b) 0 (c) 3π (d) $\frac{\pi}{2}$

19. Evaluate : $\int \frac{10x^9+10^x\log_e 10}{10^x+x^{10}}dx$

(a) $\log_e(10^x - x^{10}) + C$

(b) $\log_e(10^x + x^{10}) + C$

(c) $\log_e(10^x + x^9) + C$

(d) $\log_e(10^x - x^9) + C$

20. Evaluate : $\int \frac{1}{\sin x+\sqrt{3}\cos x}\, dx$

(a) $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$

(b) $\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+C$

(c) $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}-\frac{\pi}{6}\right)\right|+C$

(d) $\frac{1}{2}\log\left|\tan\left(x-\frac{\pi}{6}\right)\right|+C$

21. Evaluate : $\int \frac{\sec^2 x}{2+\tan x}dx$

(a) $\log|\tan x| + C$ (b) $\log|2 - \tan x| + C$

(c) $\log|2 + \tan x| + C$ (d) none of these

22. Evaluate : $\int \frac{dx}{\sqrt{1-2x-x^2}}$

(a) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$ (b) $\frac{1}{\sqrt{2}}\log(1+x)+c$

(c) $\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$ (d) $\frac{1}{\sqrt{2}}\log\left(\frac{1+x}{\sqrt{2}}\right)+c$

23. Evaluate : $\int \frac{(a^x+b^x)^2}{a^x b^x}dx$

(a) $\frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}}+\frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}}+2x+C, a\neq b$

(b) $\dfrac{\left(\dfrac{a}{b}\right)^x}{\log\dfrac{a}{b}}+\dfrac{\left(\dfrac{b}{a}\right)^x}{\log\dfrac{a}{b}}+2x+C, a\neq b$

(c) $\left(\dfrac{a}{b}\right)^x+\left(\dfrac{b}{a}\right)^x+2x+C, a\neq b$

(d) None of these

24. Find the value of $\int_{-\pi/2}^{\pi/2}|\sin x|\,dx$.

(a) 0 (b) 1 (c) 2 (d) 3

25. Evaluate : $\int_0^1\dfrac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$

(a) $\dfrac{4-\pi}{2\sqrt{2}}$ (b) $\dfrac{4+\pi}{2\sqrt{2}}$

(c) $\dfrac{4-\pi}{4\sqrt{2}}$ (d) None of these

26. Evaluate : $\int\dfrac{dx}{\sqrt{x^2-3x+2}}$

(a) $\log\left|\left(x+\dfrac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

(b) $\log\left|\left(x-\dfrac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

(c) $\log\left|\left(x-\dfrac{3}{2}\right)-\sqrt{x^2-3x+2}\right|+C$

(d) $\log\left|\left(x+\dfrac{3}{2}\right)-\sqrt{x^2-3x+2}\right|+C$

27. Evaluate : $\int\dfrac{x-4}{(x-2)^3}\cdot e^x dx$

(a) $\dfrac{e^x}{(x-2)^3}+C$ (b) $\dfrac{-e^x}{(x-2)^3}+C$

(c) $\dfrac{e^x}{(x-2)^2}+C$ (d) $\dfrac{-e^x}{(x-2)^2}+C$

28. Evaluate : $\int\dfrac{x^3-x^2+x-1}{x-1}dx$

(a) $\dfrac{x^3}{3}+x+C$ (b) x^3+x+C

(c) $\dfrac{x^3}{3}+x^2+C$ (d) $\dfrac{x^3}{3}+C$

29. Evaluate : $\int\left(5x^3+2x^{-5}-7x+\dfrac{1}{\sqrt{x}}+\dfrac{5}{x}\right)dx$

(a) $\dfrac{5x^4}{4}-\dfrac{1}{2x^4}-\dfrac{7x^2}{2}+2\sqrt{x}-5\log|x|+C$

(b) $\dfrac{5x^4}{4}-\dfrac{1}{2x^4}-\dfrac{7x^2}{2}+2\sqrt{x}+5\log|x|+C$

(c) $\dfrac{5x^4}{4}+\dfrac{1}{2x^4}+\dfrac{7x^2}{2}+2\sqrt{x}+5\log|x|+C$

(d) $\dfrac{5x^4}{4}+\dfrac{1}{2x^4}+\dfrac{7x^2}{2}+2\sqrt{x}-5\log|x|+C$

30. Evaluate : $\int\tan x\tan 2x\tan 3x\,dx$

(a) $\dfrac{1}{3}\log|\sec 3x|-\log|\sec x|+c$

(b) $\log|\sec 3x|-\dfrac{1}{2}\log|\sec 2x|+c$

(c) $\log|\sec x|-\dfrac{1}{2}\log|\sec 3x|+\dfrac{1}{2}\log|\sec 2x|+c$

(d) $\dfrac{1}{3}\log|\sec 3x|-\dfrac{1}{2}\log|\sec 2x|-\log|\sec x|+c$

31. Evaluate : $\int\dfrac{dx}{5-8x-x^2}$

(a) $\dfrac{1}{\sqrt{21}}\log\left|\dfrac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$

(b) $\dfrac{1}{2\sqrt{21}}\log\left|\dfrac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$

(c) $\dfrac{1}{\sqrt{21}}\log\left|\dfrac{\sqrt{21}-x-4}{\sqrt{21}+x+4}\right|+C$

(d) $\dfrac{1}{2\sqrt{21}}\log\left|\dfrac{\sqrt{21}-x-4}{\sqrt{21}+x+4}\right|+C$

32. Evaluate : $\int[\sin(\log x)+\cos(\log x)]dx$

(a) $x\sin(\log x)+C$ (b) $\sin(\log x)+C$

(c) $x\cos(\log x)+C$ (d) $\cos(\log x)+C$

33. Evaluate : $\int\sec^2(7-4x)dx$

(a) $\dfrac{1}{4}\tan(7-4x)+C$ (b) $\dfrac{1}{4}\tan(7+4x)+C$

(c) $\dfrac{-1}{4}\tan(7+4x)+C$ (d) $\dfrac{-1}{4}\tan(7-4x)+C$

34. Evaluate : $\int\dfrac{x^3}{x+2}dx$

(a) $\dfrac{x^3}{3}-x^2-4x-8\log|x+2|+C$

(b) $\frac{x^3}{3}-x^2+4x-8\log|x+2|+C$

(c) $\frac{x^3}{3}+x^2+4x+8\log|x+2|+C$

(d) $\frac{x^3}{3}+x^2+4x-8\log|x+2|+C$

35. Evaluate : $\int \frac{\sin x}{1+\sin x}dx$

(a) $\sec x - \tan x + C$ (b) $\sec x + \tan x + x + C$
(c) $\sec x + \tan x + C$ (d) $\sec x - \tan x + x + C$

36. Evaluate : $\int_{-\pi}^{\pi} x^{10}\sin^7 x\ dx$

(a) 1 (b) 2
(c) –1 (d) 0

37. Evaluate : $\int\left(e^{x\log a}+e^{a\log x}+e^{a\log a}\right)dx$

(a) $a^x\log a+\frac{x^{a+1}}{a+1}+\frac{a^a}{x}+c$

(b) $a^x\log a+(a+1)x^{a+1}+a^a x+c$

(c) $\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a x+c$

(d) None of these

38. Evaluate : $\int\frac{2^x+3^x}{5^x}dx$

(a) $\frac{\left(\frac{2}{5}\right)^x}{\log_e\left(\frac{2}{5}\right)}+\frac{\left(\frac{3}{5}\right)^x}{\log_e\left(\frac{3}{5}\right)}+C$

(b) $\frac{\left(\frac{2}{5}\right)^x}{\log_e\left(\frac{5}{2}\right)}+\frac{\left(\frac{3}{5}\right)^x}{\log_e\left(\frac{3}{5}\right)}+C$

(c) $\frac{\left(\frac{2}{5}\right)^x}{\log_e\left(\frac{2}{5}\right)}-\frac{\left(\frac{3}{5}\right)^x}{\log_e\left(\frac{3}{5}\right)}+C$

(d) none of these

39. Evaluate : $\int_0^2(x-[x])\,dx$

(a) 0 (b) –1
(c) 1 (d) 2

40. Evaluate : $\int\frac{(x^4-x)^{\frac{1}{4}}}{x^5}dx$

(a) $\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C$ (b) $\frac{-4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C$

(c) $\frac{2}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C$ (d) $\frac{-2}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C$

Case Based MCQs

Case I : Read the following passage and answer the questions from 41 to 45.

Integration is the process of finding the anti-derivative of a function. In this process, we are provided with the derivative of a function and asked to find out the function (*i.e.*, Primitive) Integration is the inverse process of differentiation.

Let $f(x)$ be a function of x. If there is a function $g(x)$, such that $\frac{d}{dx}(g(x)) = f(x)$, then $g(x)$ is called an integral of $f(x)$ w.r.t x and is denoted by $\int f(x)dx = g(x) + c$, where c is constant of integration.

41. $\int(3x+4)^3dx$ is equal to

(a) $\frac{(3x+4)^4}{12}+c$ (b) $\frac{3(3x+4)^4}{4}+c$

(c) $\frac{3(3x+4)^2}{2}+c$ (d) $\frac{3(3x+4)^2}{4}+c$

42. $\int\frac{(x+1)^2}{x(x^2+1)}dx$ is equal to

(a) $\log|x| + c$ (b) $\log|x|+2\tan^{-1}x+c$
(c) $-\log|x^2+1| + c$ (d) $\log|x(x^2+1)| + c$

43. $\int\sin^2 x\,dx$ is equal to

(a) $\frac{x}{2}+\frac{\sin 2x}{4}+c$ (b) $\frac{x}{2}-\frac{\sin 2x}{4}+c$

(c) $x+\frac{\sin 2x}{2}+c$ (d) $x-\frac{\sin 2x}{2}+c$

44. $\int \tan^2 x\, dx$ is equal to

(a) $\tan x + x + c$ (b) $-\tan x - x + c$

(c) $-\tan x + x + c$ (d) $\tan x - x + c$

45. $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

(a) $2\tan 2x + c$ (b) $-2\tan 2x + c$

(c) $-2\cot 2x + c$ (d) $2\cot 2x + c$

Case II : Read the following passage and answer the questions from 46 to 50.

When the intergrand can be expressed as a product of two functions, one of which can be differentiated and the other can be integrated, then we apply integration by parts.

If $f(x)$ = first function (that can be differentiated) and $g(x)$ = second function (that can be integrated), then the preference of this order can be decided by the word "*ILATE*", where

I stands for Inverse Trigonometric Function

L stands for Logarithmic Function

A stands for Algebraic Function

T stands for Trigonometric Function

E stands for Exponential Function, then

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left\{\frac{d}{dx}f(x)\int g(x)dx\right\}dx$$

46. $\int x\sin 3x\, dx =$

(a) $\frac{x\cos 3x}{3} - \frac{\sin 3x}{a} + c$

(b) $-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$

(c) $\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$

(d) $-\frac{x\cos 3x}{3} - \frac{\sin 3x}{9} + c$

47. $\int \log(x+1)\, dx =$

(a) $\log(x+1) - x + c$

(b) $x\log(x+1) - x + c$

(c) $x\log(x+1) - \log(x+1) + x + c$

(d) $x\log(x+1) + \log(x+1) - x + c$

48. $\int \tan^{-1} x\, dx =$

(a) $x\tan^{-1} x + \frac{1}{2}\log|1 - x^2| + c$

(b) $-\frac{1}{2}\log|1 + x^2| + c$

(c) $-x\tan^{-1} x - \frac{1}{2}\log|1 + x^2| + c$

(d) $x\tan^{-1} x - \frac{1}{2}\log|1 + x^2| + c$

49. $\int x^2 e^{3x}\, dx =$

(a) $\frac{e^{3x}}{9}(9x^2 + 6x + 2) + c$

(b) $\frac{e^{3x}}{9}(9x^2 - 6x + 2) + c$

(c) $\frac{e^{3x}}{27}(9x^2 + 6x + 2) + c$

(d) $\frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$

50. $\int (f(x)g''(x) - f''(x)g(x))\, dx =$

(a) $f(x)g'(x) - f'(x)g(x) + c$

(b) $f(x)g'(x) + f'(x)g(x) + c$

(c) $f'(x)g(x) - f(x)g'(x) + c$

(d) $\frac{f(x)}{g'(x)} + c$

Case III : Read the following passage and answer the questions from 51 to 55.

Let f be a continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f, then $\int_a^b f(x)dx = |f(x)|_a^b = F(b) - F(a)$

This result is very useful as it gives us a metod of calculating the definite integral easily. Here, we have no need to write integration constant c because if, we will write $F(x) + c$, instead of $f(x)$, we get

$$\int_a^b f(x)dx = |f(x) + c|_a^b = F(b) + c - F(a) - c = F(b) - F(a)$$

51. Evaluate : $\int_{\pi/4}^{\pi/2} \cos 2x\, dx$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $-\frac{1}{2}$

52. Evaluate : $\int_1^2 \frac{dx}{x^2}$

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) -1

53. $\int_{-1}^{0} \frac{dx}{2x+3}$ is equal to

(a) $\log \frac{3}{2}$ (b) $\log 3 - \log 1$

(c) $\frac{\log 3}{2}$ (d) $\log 3 + \log 1$

54. $\int_{1}^{3} (x-1)(x-2)(x-3)dx$ is equal to

(a) 3 (b) 2 (c) 1 (d) 0

55. $\int_{4}^{5} e^x dx$ equals

(a) $e^5 - e^4$ (b) $e^4 - e^5$ (c) e^9 (d) e^{20}

Assertion & Reasoning Based MCQs

Directions (Q.-56 to 60) : In these questions, a statement of Assertion is followed by a statement of Reason is given. Choose the correct answer out of the following choices :

(a) *Assertion and Reason both are correct statements and Reason is the correct explanation of Assertion.*

(b) *Assertion and Reason both are correct statements but Reason is not the correct explanation of Assertion.*

(c) *Assertion is correct statement but Reason is wrong statement.*

(d) *Assertion is wrong statement but Reason is correct statement.*

56. Assertion :

$\int \sin 3x \cos 5x \, dx = \frac{-\cos 8x}{16} + \frac{\cos 2x}{4} + C$

Reason : $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

57. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Assertion : The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x.

Reason : $\sin^2(x + \pi) = \sin^2 x$ for all real x.

58. Assertion : $I = \int_0^1 \frac{dx}{\sqrt[3]{1+x^3}} = \int_0^{2^{-1/3}} \frac{dt}{1-t^3}$

Reason : The integrand of the integral I becomes rational by the substitution $t = \frac{x}{\sqrt[3]{1+x^3}}$

59. Assertion : $\int_0^{2\pi} \sin^3 x \, dx = 0$

Reason : $\sin^3 x$ is an odd function.

60. Assertion : The value of $\int_0^{\pi/2} \sin^6 x \, dx = \frac{5\pi}{16}$.

Reason : If n is even, then $\int_0^{\pi/2} \sin^n x \, dx$ equals $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}$.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (VSA)

1. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

2. Evaluate : $\int \cos^{-1}(\sin x)dx$

3. Write the value of $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$.

4. Write the value of $\int \frac{2 - 3\sin x}{\cos^2 x} dx$.

5. Evaluate : $\int \frac{(\log x)^2}{x} dx$

6. Find : $\int \frac{dx}{9 + 4x^2}$

7. Find : $\int x^4 \log x \, dx$

8. If $\int_0^a \frac{1}{4 + x^2} dx = \frac{\pi}{8}$, find the value of a.

9. Write the value of $\int_0^1 \frac{e^x}{1 + e^{2x}} dx$.

10. Find the value of $\int_1^4 |x - 5| dx$.

Short Answer Type Questions (SA-I)

11. Find : $\int \sqrt{1-\sin 2x}\,dx, \frac{\pi}{4} < x < \frac{\pi}{2}$

12. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

13. Find : $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

14. Find : $\int \frac{dx}{x^2+4x+8}$

15. Find $\int \frac{x+1}{(x+2)(x+3)} dx.$

16. Find : $\int \sin^{-1}(2x) dx$

17. Find : $\int x \cdot \tan^{-1} x\, dx$

18. Evaluate $\int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx.$

19. Find the value of $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) dx.$

20. Find : $\int_{-\frac{\pi}{4}}^{0} \frac{1+\tan x}{1-\tan x} dx$

Short Answer Type Questions (SA-II)

21. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

22. Evaluate : $\int \sin x \sin 2x \sin 3x\, dx$

23. Find $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx.$

24. Evaluate : $\int \frac{x+2}{\sqrt{x^2+5x+6}} dx$

25. Evaluate : $\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx$

26. Evaluate : $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$

27. Find : $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$

28. Find : $\int \frac{x}{(x^2+1)(x-1)} dx$

29. Evaluate : $\int e^{2x} \cdot \sin(3x+1) dx$

30. Evaluate : $\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}} dx$

31. Evaluate : $\int_0^{\pi/2} x^2 \sin x\, dx$

32. Evaluate $\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4}+x\right) dx.$

33. Evaluate : $\int_4^{\pi} \frac{x}{1+\sin\alpha \sin x} dx$

34. Evaluate : $\int_1^4 [\,|x-1|+|x-2|+|x-3|\,]\, dx$

35. Using properties of definite integrals, evaluate the following : $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} dx$

Long Answer Type Questions (LA)

36. Evaluate : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx$

37. Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$

38. Find : $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx, x \in [0,1]$

39. Prove that

$$\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2} \cdot \frac{\pi}{2}$$

40. Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$

ANSWERS

OBJECTIVE TYPE QUESTIONS

1. **(a) :** Let

$I = \int (3\sin x - 2\cos x + 4\sec^2 x - 5\,\text{cosec}^2 x)\, dx$

$\Rightarrow \quad I = 3\int \sin x\, dx - 2\int \cos x\, dx + 4\int \sec^2 x\, dx - 5\int \text{cosec}^2 x\, dx$

$\Rightarrow \quad I = -3\cos x - 2\sin x + 4\tan x + 5\cot x + C$

2. **(b) :** We have, $\int (2^x + 2^{-x})^2\, dx = \int (2^{2x} + 2^{-2x} + 2)\, dx$

$= \dfrac{2^{2x}}{(\log 2)\times 2} + \dfrac{2^{-2x}}{(\log 2)(-2)} + 2\cdot x + C$

$= \dfrac{1}{2\log 2}(2^{2x} - 2^{-2x}) + 2x + C$

3. **(c) :** We have, $\int \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int (\sec^2 x - \text{cosec}^2 x) dx$

$= \tan x + \cot x + C$

4. **(a) :** Let $I = \int_2^4 \dfrac{x}{x^2+1} dx$

Put $x^2 + 1 = t \quad \Rightarrow \quad 2x dx = dt \quad \Rightarrow \quad x\, dx = \dfrac{1}{2} dt$

Also, $x = 2 \Rightarrow t = 5$ and $x = 4 \Rightarrow t = 17$

$\therefore \quad I = \dfrac{1}{2}\int_5^{17} \dfrac{dt}{t} = \dfrac{1}{2}[\log t]_5^{17} = \dfrac{1}{2}[\log 17 - \log 5] = \dfrac{1}{2}\log\left(\dfrac{17}{5}\right)$

5. **(b) :** Let $I = \int x e^{x^2} dx$

Put $x^2 = t \quad \Rightarrow \quad 2x dx = dt \quad \Rightarrow \quad x\, dx = \dfrac{dt}{2}$

$\therefore \quad I = \dfrac{1}{2}\int e^t dt = \dfrac{e^t}{2} + C = \dfrac{e^{x^2}}{2} + C$

6. **(c) :** We have,

$\int \dfrac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3} dx = \int \dfrac{\cos^2(x/2) - \sin^2(x/2)}{\{\cos(x/2) + \sin(x/2)\}^3} dx$

Put $t = \cos\dfrac{x}{2} + \sin\dfrac{x}{2} \quad \Rightarrow 2dt = \left[\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right] dx$

$\Rightarrow \int \dfrac{\cos(x/2) - \sin(x/2)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} dx = 2\int \dfrac{1}{t^2} dt$

$= \dfrac{-2}{t} + C = \dfrac{-2}{\cos(x/2) + \sin(x/2)} + C$

7. **(d) :** We have, $\int_2^4 \dfrac{(x^2+x)}{\sqrt{2x+1}} dx$

Integrating by parts, we get

$\int_2^4 \dfrac{(x^2+x)}{\sqrt{2x+1}} dx = \left[(x^2+x)\cdot\sqrt{2x+1}\right]_2^4 - \int_2^4 (2x+1)\cdot\sqrt{2x+1}\, dx$

$= (60 - 6\sqrt{5}) - \int_2^4 (2x+1)^{3/2} dx$

$= (60 - 6\sqrt{5}) - \dfrac{1}{5}\cdot\left[(2x+1)^{5/2}\right]_2^4$

$= (60 - 6\sqrt{5}) - \left(\dfrac{243}{5} - 5\sqrt{5}\right) = \left(\dfrac{57}{5} - \sqrt{5}\right) = \left(\dfrac{57 - 5\sqrt{5}}{5}\right)$

8. **(a) :** Let $I = \int 2^{2^{2^x}} 2^{2^x} 2^x dx$

Let $2^{2^{2^x}} = t \quad \Rightarrow \quad 2^{2^{2^x}} 2^{2^x} 2^x (\log 2)^3 dx = dt$

$\Rightarrow \quad I = \int \dfrac{1}{(\log 2)^3} dt = \dfrac{1}{(\log 2)^3} t + C = \dfrac{1}{(\log 2)^3} 2^{2^{2^x}} + C$

9. **(c) :** $\int 2^{(x+3)} dx = \int 2^x \cdot 2^3 dx = 8\int 2^x dx$

$= 8\cdot\dfrac{2^x}{\log 2} + C \quad = \dfrac{2^{(x+3)}}{\log 2} + C$

10. **(c) :** Let $I = \int \sin^3 x \cos^3 x\, dx$

$\Rightarrow \quad I = \dfrac{1}{8}\int (2\sin x\cos x)^3 dx$

$\Rightarrow \quad I = \dfrac{1}{8}\int \sin^3 2x\, dx \Rightarrow \quad I = \dfrac{1}{8}\int \dfrac{3\sin 2x - \sin 6x}{4} dx$

$\Rightarrow \quad I = \dfrac{1}{32}\left\{-\dfrac{3}{2}\cos 2x + \dfrac{1}{6}\cos 6x\right\} + C$

11. **(b) :** Let $I = \int \sqrt{(x-3)(5-x)}\, dx = \int \sqrt{-x^2 + 8x - 15}\, dx$

$\Rightarrow \quad I = \int \sqrt{-\{x^2 - 8x + 16 - 16 + 15\}}\, dx$

$\Rightarrow \quad I = \int \sqrt{-\{(x-4)^2 - 1^2\}}\, dx = \int \sqrt{1^2 - (x-4)^2}\, dx$

$\Rightarrow \quad I = \dfrac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \dfrac{1}{2}\sin^{-1}\left(\dfrac{x-4}{1}\right) + C$

12. **(c) :** Let $I = \int_0^{\pi/4} \tan^3 x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\, dx$

$= \int_0^{\pi/4} \sec^2 x \tan x\, dx - \int_0^{\pi/4} \tan x\, dx$

Put $\tan x = t$ in first integral $\Rightarrow \sec^2 x\, dx = dt$

When $x = 0 \quad \Rightarrow \quad t = 0$

$x = \pi/4 \quad \Rightarrow \quad t = 1$

$\therefore \quad I = \int_0^1 t\, dt - \int_0^{\pi/4} \tan x\, dx = \left[\dfrac{t^2}{2}\right]_0^1 - [\log|\sec x|]_0^{\pi/4}$

$$=\left(\frac{1}{2}-0\right)-\log\left|\sec\frac{\pi}{4}\right|+\log|\sec 0|=\frac{1}{2}(1-\log 2)$$

13. **(a)** : Let $I=\int\frac{\cot x}{\sqrt[3]{\sin x}}dx=\int\frac{\cos x}{\sin^{1/3}x\cdot\sin x}dx$

$$=\int\frac{\cos x}{\sin^{4/3}x}dx=\int\sin^{-4/3}x\cdot\cos x dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\Rightarrow\ I=\int t^{-4/3}\,dt=\frac{t^{-1/3}}{-1/3}+C=\frac{-3}{\sqrt[3]{\sin x}}+C$$

14. **(a)** : Let $I=\int\frac{x^2}{(ax+b)^2}dx$

Put $ax+b=t\ \Rightarrow\ dx=\frac{1}{a}dt$

$$\therefore\ I=\frac{1}{a^3}\int\frac{(t-b)^2}{t^2}dt=\frac{1}{a^3}\int\left(1+\frac{b^2}{t^2}-\frac{2b}{t}\right)dt$$

$$=\frac{1}{a^3}\left(t-\frac{b^2}{t}-2b\log t\right)+C$$

$$=\frac{1}{a^3}\left(ax+b-\frac{b^2}{ax+b}-2b\log(ax+b)\right)+C$$

15. **(b)** : We have, $\int_0^1\left\{e^x+\sin\frac{\pi x}{4}\right\}dx$

$$=\left[e^x\right]_0^1+\frac{4}{\pi}\left[-\cos\frac{\pi}{4}x\right]_0^1=e-1-\frac{4}{\sqrt{2}\pi}+\frac{4}{\pi}$$

$$=e-1-\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$$

16. **(b)** : Let $I=\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

Put $x^{3/2}=t\ \Rightarrow\ \frac{3}{2}x^{1/2}dx=dt$

$$\therefore\ I=\frac{2}{3}\int\frac{dt}{\sqrt{a^3-t^2}}=\frac{2}{3}\int\frac{dt}{\sqrt{(a^{3/2})^2-t^2}}$$

$$=\frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right]+c=\frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right]+c$$

$$=\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+c$$

17. **(d)** : We have, $\int_0^2 e^{3-4x}dx=\left[\frac{e^{3-4x}}{-4}\right]_0^2$

$$=-\frac{1}{4}\left[e^{3-8}-e^{3-0}\right]=\frac{-1}{4}\left[e^{-5}-e^{3}\right]$$

18. **(a)** : Let $I=\int_0^{2\pi}\frac{dx}{e^{\sin x}+1}$ ...(i)

$$\Rightarrow\ I=\int_0^{2\pi}\frac{dx}{e^{\sin(2\pi-x)}+1}\qquad\left(\because\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow\ I=\int_0^{2\pi}\frac{dx}{e^{-\sin x}+1}\ \Rightarrow\ I=\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+1}dx\qquad\text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{2\pi}1\cdot dx=2\pi\quad\therefore\quad I=\pi$$

19. **(b)** : Let $I=\int\frac{10x^9+10^x\log_e 10}{10^x+x^{10}}dx$

Put $10^x+x^{10}=t$

$\Rightarrow\ (10^x\log_e 10+10x^9)dx=dt$

$$\therefore\ I=\int\frac{10x^9+10^x\log_e 10}{10^x+x^{10}}dx=\int\frac{dt}{t}$$

$=\log_e t+C\ =\log_e(10^x+x^{10})+C.$

20. **(a)** : Let $I=\int\frac{1}{\sin x+\sqrt{3}\cos x}dx$

$$=\frac{1}{2}\int\frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$$

$$\Rightarrow\ I=\frac{1}{2}\int\frac{1}{\sin\left(x+\frac{\pi}{3}\right)}dx=\frac{1}{2}\int\text{cosec}\left(x+\frac{\pi}{3}\right)dx$$

$$\Rightarrow\ I=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$$

$$\left[\because\int\text{cosec}x\,dx=\log\left|\tan\frac{x}{2}\right|+C\right]$$

21. **(c)** : Let $I=\int\frac{\sec^2 x}{2+\tan x}dx$

Put $2+\tan x=t\ \Rightarrow\ \sec^2 x\,dx=dt$

$$\therefore\ I=\int\frac{dt}{t}=\log|t|+C=\log|2+\tan x|+C$$

22. **(c)** : Let $I=\int\frac{dx}{\sqrt{1-(x^2+2x)}}=\int\frac{dx}{\sqrt{2-(x^2+2x+1)}}$

$$=\int\frac{dx}{\sqrt{2-(1+x)^2}}=\int\frac{dx}{\sqrt{(\sqrt{2})^2-(1+x)^2}}$$

Put $1+x=z\ \Rightarrow dx=dz$

$$\therefore\ I=\int\frac{dz}{\sqrt{(\sqrt{2})^2-z^2}}=\sin^{-1}\frac{z}{\sqrt{2}}+c\ =\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$$

23. **(a)** : We have, $\int\frac{(a^x+b^x)^2}{a^xb^x}dx=\int\frac{a^{2x}+b^{2x}+2a^xb^x}{a^xb^x}dx$

$$=\int\left(\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x+2\right)dx=\frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}}+\frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}}+2x+C,\ a\neq b$$

24. (c) : $\because$ $|\sin x|$ is an even function.

$\therefore \int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2\int_0^{\pi/2} |\sin x|\,dx = 2\int_0^{\pi/2} \sin x\,dx$

$= -2[\cos x]_0^{\pi/2} = -2(0-1) = 2$

25. (c) : Let $\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$

Put $\tan^{-1}x = \theta \Rightarrow x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

When, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \quad I = \int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx = \int_0^{\pi/4} \frac{\theta\tan\theta}{\sec^3\theta}\sec^2\theta\,d\theta$

$= \int_0^{\pi/4} \theta\sin\theta\,d\theta = [-\theta\cos\theta]_0^{\pi/4} - \int_0^{\pi/4}(-\cos\theta)\,d\theta$

[Integrating by parts]

$= [-\theta\cos\theta]_0^{\pi/4} + [\sin\theta]_0^{\pi/4} = \frac{4-\pi}{4\sqrt{2}}$

26. (b) : We have, $\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{\left(x^2-3x+\frac{9}{4}\right)-\frac{1}{4}}}$

$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}} = \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + C$

27. (c) : $\int \frac{x-4}{(x-2)^3}\cdot e^x dx = \int\left[\frac{x-2}{(x-2)^3} - \frac{2}{(x-2)^3}\right]e^x dx$

$= \int\left[\frac{1}{(x-2)^2} - \frac{2}{(x-2)^3}\right]e^x dx = \frac{e^x}{(x-2)^2} + C$

$\left[\because \int[f(x)+f'(x)]e^x dx = e^x f(x) + C\right]$

28. (a) : Let $I = \int \frac{x^3-x^2+x-1}{x-1}dx$

$= \int \frac{x^2(x-1)+1(x-1)}{x-1} = \int \frac{(x^2+1)(x-1)}{x-1}dx$

$= \int (x^2+1)dx = \frac{1}{3}x^3 + x + C$

29. (b) : We have $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$= 5\int x^3dx + 2\int x^{-5}dx - 7\int x\,dx + \int x^{-1/2}dx + 5\int \frac{1}{x}dx$

$= 5\cdot\frac{x^4}{4} + 2\cdot\frac{x^{-4}}{(-4)} - 7\cdot\frac{x^2}{2} + \frac{x^{1/2}}{(1/2)} + 5\log|x| + C$

$= \frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5\log|x| + C$

30. (d) : Let $I = \int \tan x\tan 2x\tan 3x\,dx$

Since, $\tan 3x = \tan(2x+x) = \frac{\tan 2x + \tan x}{1-\tan x\tan 2x}$

$\Rightarrow \tan x\tan 2x\tan 3x = \tan 3x - \tan 2x - \tan x$ ...(i)

$\therefore \quad I = \int(\tan 3x - \tan 2x - \tan x)dx$ (From (i))

$= \frac{1}{3}\log|\sec 3x| - \frac{1}{2}\log|\sec 2x| - \log|\sec x| + c$

31. (b) : Let $I = \int \frac{dx}{5-8x-x^2} = \int \frac{dx}{21-(x+4)^2}$

$= \int \frac{dx}{(\sqrt{21})^2-(x+4)^2} = \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$

32. (a) : Let $I = \int[\sin(\log x)+\cos(\log x)]dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$\therefore \quad I = \int(\sin t + \cos t)e^t dt = e^t\sin t + C$

$= x\sin(\log x) + C$

$[\because \int[f(x) + f'(x)]\,e^x\,dx = e^x f(x) + C)$

33. (d) : Let $I = \int \sec^2(7-4x)dx$

Put $7 - 4x = t \Rightarrow dx = \frac{-1}{4}dt$

$\therefore \quad I = \int \frac{\sec^2 t}{-4}dt \Rightarrow I = \frac{\tan t}{-4} + C = \frac{\tan(7-4x)}{-4} + C$

34. (b) : Let $I = \int \frac{x^3}{x+2}dx$

Dividing x^3 by $x + 2$, we get

$= \int\left(x^2-2x+4-\frac{8}{x+2}\right)dx$

$= \frac{x^3}{3} - x^2 + 4x - 8\log|x+2| + C$

35. (d) : Let $I = \int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$= \int \frac{\sin x - \sin^2 x}{\cos^2 x}dx = \int \sec x\tan x\,dx - \int \tan^2 x\,dx$

$= \int \sec x\tan x\,dx - \int(\sec^2 x - 1)dx = \sec x - \tan x + x + C$

36. (d) : Let $I = \int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx$

Let $f(x) = x^{10}\sin^7 x$

and $f(-x) = (-x)^{10}[\sin(-x)]^7 = -x^{10}\sin^7 x = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore \quad I = \int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx = 0$

37. (c) : Let $I = \int\left(e^{x\log a} + e^{a\log x} + e^{a\log a}\right)dx$

$= \int\left(e^{\log a^x} + e^{\log x^a} + e^{\log a^a}\right)dx = \int\left(a^x + x^a + a^a\right)dx$

$[\because e^{\log y} = y]$

$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a x + c$

38. **(a) :** Let $I=\int\frac{2^x+3^x}{5^x}dx$

$\Rightarrow \quad I=\int\frac{2^x}{5^x}dx+\int\frac{3^x}{5^x}dx=\int\left(\frac{2}{5}\right)^x dx+\int\left(\frac{3}{5}\right)^x dx$

$\Rightarrow \quad I=\frac{\left(\frac{2}{5}\right)^x}{\log_e\left(\frac{2}{5}\right)}+\frac{\left(\frac{3}{5}\right)^x}{\log_e\left(\frac{3}{5}\right)}+C$

39. **(c) :** Let $I=\int_0^2(x-[x])dx=\int_0^2 x\,dx-\int_0^2[x]dx$

$=\left[\frac{x^2}{2}\right]_0^2-\int_0^1[x]dx-\int_1^2[x]dx=\frac{4}{2}-\int_0^1 0\,dx-\int_1^2 1\,dx$

$= 2 - 0 - [x]_1^2 = 2 - [2 - 1] = 2 - 1 = 1.$

40. **(a) :** Let $I=\int\frac{(x^4-x)^{\frac{1}{4}}}{x^5}dx$

$\Rightarrow \quad I=\int\frac{x\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}}}{x^5}dx=\int\frac{\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}}}{x^4}dx$

Put $1-\frac{1}{x^3}=t\Rightarrow\frac{3}{x^4}dx=dt$

$\therefore \quad I=\frac{1}{3}\int t^{\frac{1}{4}}dt=\frac{1}{3}\times\frac{4}{5}t^{\frac{5}{4}}+C=\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C.$

41. **(a) :** $\int(3x+4)^3dx=\frac{(3x+4)^4}{4\cdot 3}+c=\frac{(3x+4)^4}{12}+c$

42. **(b) :** Let $I=\int\frac{(x+1)^2}{x(x^2+1)}dx=\int\frac{x^2+1+2x}{x(x^2+1)}dx$

$=\int\left(\frac{1}{x}+\frac{2}{x^2+1}\right)dx=\log|x|+2\tan^{-1}x+c$

43. **(b) :** $\int\sin^2 x\,dx=\int\frac{1-\cos 2x}{2}dx$

$=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c=\frac{x}{2}-\frac{\sin 2x}{4}+c$

44. **(d) :** $\int\tan^2 x\,dx=\int(\sec^2 x-1)\,dx$

$=\tan x-x+c$

45. **(c) :** Let $I=\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{4}{4\sin^2 x\cos^2 x}dx$

$=4\int\text{cosec}^2 2x\,dx=-2\cot 2x+c$

46. **(b) :** Let $I=\int\underset{\text{I}}{x}\,\underset{\text{II}}{\sin 3x}\,dx$

$=x\int\sin 3x\,dx-\int\left(\frac{d}{dx}(x).\int\sin 3x\,dx\right)dx$

$=x\left(-\frac{\cos 3x}{3}\right)-\int 1\cdot\left(-\frac{\cos 3x}{3}\right)dx+c$

$=-\frac{x\cos 3x}{3}+\frac{1}{3}\int\cos 3x\,dx+c$

$\therefore\ I=-\frac{x\cos 3x}{3}+\frac{1}{3}\cdot\frac{\sin 3x}{3}+c=-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}+c$

47. **(d) :** Let $I=\int\log(x+1)\,dx=\int\underset{\text{I}}{\log(x+1)}\cdot\underset{\text{II}}{1}dx$

$=\log(x+1)\cdot x-\int\frac{1}{x+1}\cdot x\,dx$

$=x\log(x+1)-\int\frac{x+1}{x+1}dx+\int\frac{1}{x+1}dx$

$=x\log(x+1)-x+\log(x+1)+c$

48. **(d) :** Let $I=\int\tan^{-1}x\,dx=\int\underset{\text{I}}{\tan^{-1}x}\cdot\underset{\text{II}}{1}\,dx$

$=\tan^{-1}x\int 1\,dx-\int\left[\frac{d}{dx}(\tan^{-1}x)\int 1\,dx\right]dx$

$=x\tan^{-1}x-\int\frac{1}{1+x^2}(x)\,dx$

$=x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx$

$\therefore \quad I=x\tan^{-1}x-\frac{1}{2}\log|1+x^2|+c$

49. **(d) :** Let $I=\int\underset{\text{I}}{x^2}\underset{\text{II}}{e^{3x}}\,dx$

$=x^2\left(\frac{e^{3x}}{3}\right)-\int 2x\frac{e^{3x}}{3}dx$

$=\frac{x^2e^{3x}}{3}-(2x)\left(\frac{e^{3x}}{9}\right)+(2)\left(\frac{e^{3x}}{27}\right)+c$

$\therefore \quad I=\frac{e^{3x}}{27}(9x^2-6x+2)+c$

50. **(a) :** Let $I=\int(f(x)g''(x)-f''(x)g(x))\,dx$

$=\int\underset{\text{I}}{f(x)}\underset{\text{II}}{g''(x)}\,dx-\int\underset{\text{I}}{g(x)}\underset{\text{II}}{f''(x)}\,dx$

$=f(x)g'(x)-\int f'(x)g'(x)\,dx-g(x)f'(x)+\int g'(x)f'(x)\,dx$

$=f(x)g'(x)-g(x)f'(x)+c$

51. **(d) :** We have, $I=\int_{\pi/4}^{\pi/2}\cos 2x\,dx$

$=\left[\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}=-\frac{1}{2}$

52. **(a) :** Let $I=\int_1^2\frac{dx}{x^2}\Rightarrow I=\left[\left(\frac{-1}{x}\right)\right]_1^2=\frac{1}{2}$

53. **(c) :** We have $I=\int_{-1}^{0}\frac{dx}{2x+3}$

$=\left[\frac{\log(2x+3)}{2}\right]_{-1}^{0}=\left[\frac{\log 3}{2}-\frac{\log 1}{2}\right]=\frac{\log 3}{2}$

54. **(d) :** We have, $\int_1^3 (x-1)(x-2)(x-3)dx$

$= \int_1^3 (x^3 - 6x^2 + 11x - 6)dx = \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{11x^2}{2} - 6x\right]_1^3$

$= \left[\frac{81}{4} - \frac{162}{3} + \frac{99}{2} - 18 - \left(\frac{1}{4} - \frac{6}{3} + \frac{11}{2} - 6\right)\right] = 0$

55. **(a) :** We have, $\int_4^5 e^x dx = [e^x]_4^5 = e^5 - e^4$

56. **(a) :** We have, $\int \sin 3x \cos 5x \, dx$

$= \frac{1}{2}\int 2\cos 5x \sin 3x \, dx$

$= \frac{1}{2}\int (\sin 8x - \sin 2x)dx = \frac{1}{2}\left[\int \sin 8x \, dx - \int \sin 2x \, dx\right]$

$= \frac{1}{2}\left[\frac{-\cos 8x}{8}\right] - \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + C = \frac{-\cos 8x}{16} + \frac{\cos 2x}{4} + C$

$\therefore$ Both assertion and reason are true and reason is the correct explanation of assertion.

57. **(d) :** $F(x) = \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)dx$

$= \frac{x}{2} - \frac{\sin 2x}{4} + C$

$\because$ $F(x+\pi) - F(x) = \frac{\pi}{2} \neq 0$

$\therefore$ Assertion is false.

$\sin^2(x+\pi) = (-\sin x)^2 = \sin^2 x$

$\therefore$ Reason is true.

58. **(a) :** Let $t = \frac{x}{\sqrt[3]{1+x^3}} \Rightarrow dt = \frac{dx}{(1+x^3)^{\frac{4}{3}}}$

$\therefore$ $(1+x^3)t^3 = x^3 \Rightarrow t^3 + x^3t^3 = x^3$

$\Rightarrow$ $t^3 = x^3(1-t^3) \Rightarrow x^3 = \frac{t^3}{1-t^3}$

$\Rightarrow 1 + x^3 = \frac{1}{1-t^3}$

When $x = 0$, $t = 0$ and $x = 1$, $t = 2^{-1/3}$

$\Rightarrow$ $I = \int_0^{2^{-1/3}} \frac{dt}{1-t^3}$

59. **(b) :** Let $I = \int_0^{2\pi} \sin^3 x \, dx = \int_0^{2\pi} (1-\cos^2 x)\sin x \, dx$

Putting $\cos x = t \Rightarrow \sin x \, dx = -dt$

When $x = 0$, $t = 1$ and $x = 2\pi$, $t = 1$

$\therefore$ $I = \int_1^1 (1-t^2)(-dt) = 0$

60. **(d) :** Reason is obvious.

$\therefore$ $\int_0^{\pi/2} \sin^6 x \, dx = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{32}$

$\therefore$ Assertion is false.

SUBJECTIVE TYPE QUESTIONS

1. The antiderivative of $3\sqrt{x} + \frac{1}{\sqrt{x}}$

$= \int\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = 3\int x^{1/2}dx + \int x^{-1/2}dx$

$= 3 \cdot \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = 2x\sqrt{x} + 2\sqrt{x} + C$

$= 2\sqrt{x}(x+1) + C$

2. $\int \cos^{-1}(\sin x)dx = \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]dx$

$= \int\left(\frac{\pi}{2} - x\right)dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$

3. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \tan^2 x \, dx$

$= \int (\sec^2 x - 1)dx = \tan x - x + C$

4. $\int \frac{2 - 3\sin x}{\cos^2 x}dx = \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$

$= \int (2\sec^2 x - 3\sec x \tan x)dx = 2\tan x - 3\sec x + C$

5. Let $I = \int \frac{(\log x)^2}{x}dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore$ $I = \int t^2 dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$

6. Let $I = \int \frac{dx}{9+4x^2} = \frac{1}{4}\int \frac{dx}{x^2 + \frac{9}{4}} = \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2}$

$= \frac{1}{4} \cdot \frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C$

7. Let $I = \int x^4 \log x \, dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5}dx$

[Integrating by parts]

$= \frac{x^5}{5}\log x - \frac{1}{5}\int x^4 dx = \frac{1}{5}x^5 \log x - \frac{x^5}{25} + C$

8. Here, $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$

$\Rightarrow \int_0^a \frac{1}{x^2 + 2^2}dx = \frac{\pi}{8} \Rightarrow \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8} \Rightarrow \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$

$\Rightarrow \frac{a}{2} = \tan\frac{\pi}{4} = 1 \Rightarrow a = 2$

9. Let $I = \int_0^1 \frac{e^x}{1+e^{2x}} dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$
Also, $x = 0 \Rightarrow t = e^0 = 1$
and $x = 1 \Rightarrow t = e^1 = e$

$\therefore \quad I = \int_1^e \frac{dt}{(1+t^2)} = [\tan^{-1} t]_1^e = \tan^{-1} e - \tan^{-1} 1$

$= \tan^{-1}\left(\frac{e-1}{1+e}\right)$

10. Let $I = \int_1^4 |x-5|\, dx$

$= -\int_1^4 (x-5)\, dx = \left[-\frac{x^2}{2} + 5x\right]_1^4$

$= -\frac{16}{2} + 5\cdot 4 + \frac{1}{2} - 5 = -8 + 20 - 5 + \frac{1}{2} = 7 + \frac{1}{2} = \frac{15}{2}$

11. Let $I = \int (\sqrt{1-\sin 2x})\, dx$

$= \int \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\, dx$

$= \pm \int (\cos x - \sin x)\, dx$

Since, $\frac{\pi}{4} < x < \frac{\pi}{2}$, so we get

$I = \int (\sin x - \cos x) dx = -(\cos x + \sin x) + C$

12. Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$= \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x} dx$

$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx$

$= \int \sec^2 x\, dx = \tan x + C$

13. Let $I = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - 2x - x^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - 1 - 2x - x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}}$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$

14. We have, $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4+4}$

$= \int \frac{dx}{(x+2)^2 + (2)^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

15. Let $I = \int \frac{(x+1)}{(x+2)(x+3)} dx$

Also let, $\frac{(x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$

$\Rightarrow \quad x + 1 = A(x + 3) + B(x + 2)$ ...(i)

Putting $x = -3$ in (i), we get
$-B = -3 + 1 = -2 \Rightarrow B = 2$
Putting $x = -2$ in (i), we get
$A = -2 + 1 = -1$

$\therefore \quad I = \int \frac{-1}{(x+2)} dx + 2\int \frac{1}{(x+3)} dx$

$= -\log(x + 2) + 2\log(x + 3) + C$

16. Let $I = \int \sin^{-1}(2x)\, dx = \int 1\cdot \sin^{-1}(2x)\, dx$

Integrating by parts, we get

$= \sin^{-1}(2x)x - \int \left(\frac{1}{\sqrt{1-4x^2}} \frac{d}{dx}(2x)\cdot x\right) dx$

$= x\sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} dx$

$= x\sin^{-1}(2x) + \int \frac{dt}{4\sqrt{t}}$

(Putting $1 - 4x^2 = t \Rightarrow -8x dx = dt$)

$= x\sin^{-1}(2x) + \frac{2}{4}(t)^{1/2} + C$

$= x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

17. Let $I = \int x\cdot \tan^{-1} x\, dx$

Integrating by parts, we get

$I = \tan^{-1} x \int x dx - \int \left\{\frac{d}{dx}(\tan^{-1} x)\int x dx\right\} dx$

$= (\tan^{-1} x)\frac{x^2}{2} - \int \frac{1}{(1+x^2)} \frac{x^2}{2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$

$= \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{x}{2} + C$

18. Let $I = \int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx$

Putting $2x = y \Rightarrow 2dx = dy$

As $x \to 1 \Rightarrow y \to 2$ and $x \to 2 \Rightarrow y \to 4$

$$\therefore \quad I = \frac{1}{2}\int_2^4\left[\frac{2}{y}-\frac{2}{y^2}\right]e^y dy = \int_2^4\left[\frac{1}{y}-\frac{1}{y^2}\right]e^y dy$$

$$= \left[e^y \cdot \frac{1}{y}\right]_2^4 = \frac{1}{4}e^4 - \frac{1}{2}e^2 = \frac{e^2}{2}\left(\frac{e^2}{2}-1\right)$$

19. Let $I = \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$

$$= \int_0^1 \tan^{-1}\left[\frac{(1-x)-x}{1+x(1-x)}\right]dx$$

$$I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}x]\,dx \qquad \text{...(i)}$$

$$I = \int_0^1 [\tan^{-1}x - \tan^{-1}(1-x)]\,dx \qquad \text{...(ii)}$$

$$\left[\text{Using property, } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}x + \tan^{-1}x - \tan^{-1}(1-x)]\,dx = 0$$

$\Rightarrow \quad I = 0$

20. Let $I = \int_{-\frac{\pi}{4}}^{0} \frac{(1+\tan x)}{(1-\tan x)}dx = \int_{-\frac{\pi}{4}}^{0} \frac{\left(1+\frac{\sin x}{\cos x}\right)}{\left(1-\frac{\sin x}{\cos x}\right)}dx$

$$= \int_{-\frac{\pi}{4}}^{0} \frac{\cos x + \sin x}{\cos x - \sin x}dx$$

Put $\cos x - \sin x = t \Rightarrow -(\sin x + \cos x)\,dx = dt$

When $x = 0, t = 1$, when $x = \frac{-\pi}{4}, t = \sqrt{2}$

$$\therefore \quad I = \int_{\sqrt{2}}^{1} -\frac{dt}{t} = \int_1^{\sqrt{2}} \frac{dt}{t} = [\log t]_1^{\sqrt{2}}$$

$$= \log\sqrt{2} - \log 1 = \frac{1}{2}\log 2$$

21. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx = \int \frac{\sin(x+a-2a)}{\sin(x+a)}dx$

$$= \int\left(\frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)}\right)dx$$

$$\Rightarrow \quad I = \cos 2a\int dx - \sin 2a\int \frac{\cos(x+a)}{\sin(x+a)}dx$$

Put $\sin(x+a) = t \quad \Rightarrow \quad \cos(x+a)dx = dt$

$$\Rightarrow \quad I = \cos 2a\int dx - \sin 2a\int\frac{dt}{t}$$

$= x\cos 2a - \sin 2a\log|\sin(x+a)| + C$

22. $\int \sin x \sin 2x \sin 3x\; dx$

$$= \int \sin 3x \sin x \sin 2x\; dx$$

$$= \frac{1}{2}\int (\cos 2x - \cos 4x)\sin 2x\; dx$$

$$= \frac{1}{2}\int \sin 2x\cos 2x\; dx - \frac{1}{2}\int \cos 4x\sin 2x\; dx$$

$$= \frac{1}{4}\int \sin 4x\; dx - \frac{1}{4}\int (\sin 6x - \sin 2x)\; dx$$

$$= \frac{1}{4}\int \sin 4x\; dx - \frac{1}{4}\int \sin 6x\; dx + \frac{1}{4}\int \sin 2x\; dx$$

$$= \frac{1}{4}\left[\frac{-\cos 4x}{4} - \frac{(-\cos 6x)}{6} + \frac{(-\cos 2x)}{2}\right] + C$$

$$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + C$$

23. Let $I = \int(\sqrt{\tan x} + \sqrt{\cot x})dx$

$$= \int\left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right)dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{2\sin x\cos x}}dx = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin 2x + 1 - 1}}dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(1-\sin 2x)}}dx = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

$$\therefore \quad I = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\sin^{-1}t + C$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

24. Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx$

$$= \frac{1}{2}\int (x^2+5x+6)^{-1/2}(2x+5)dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

Put $x^2 + 5x + 6 = t \quad \Rightarrow \quad (2x+5)\,dx = dt$

$$\Rightarrow \quad I = \frac{1}{2}\int t^{-1/2}dt - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} + C$$

$$= \frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x + \frac{5}{2} + \sqrt{x^2+5x+6}\right| + C$$

25. Let $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}}dx$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$= I_1 + I_2$ (say) ...(1)

where $I_1 = \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Put $x^2 + 4x + 10 = t \Rightarrow (2x + 4)dx = dt$

$\therefore \quad I_1 = \frac{5}{2}\int t^{-1/2}dt = \frac{5}{2}\cdot\frac{t^{1/2}}{(1/2)} = 5\sqrt{t}$

$= 5\sqrt{x^2+4x+10} + C_1$ ...(2)

and $I_2 = -7\int \frac{dx}{\sqrt{x^2+4x+10}}$

$= -7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$

$= -7\log|x+2+\sqrt{x^2+4x+10}| + C_2$ ...(3)

From (1), (2) and (3), we get

$I = 5\sqrt{x^2+4x+10} - 7\log|x+2+\sqrt{x^2+4x+10}| + C,$

where $C = C_1 + C_2$

26. Let $I = \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}dx = \int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x - 1}}dx$

$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}}dx$

$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}}\cdot dx$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$\therefore \quad I = \int \frac{-dt}{\sqrt{t^2-1}} = -\log|t + \sqrt{t^2-1}| + C$

(where $t = \sin x + \cos x$)

$= -\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$

27. Let $I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Let $x^2 = t$

$\therefore \quad \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{(t+1)(t+4)}{(t+3)(t-5)}$

$= \frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{7t+19}{(t+3)(t-5)}$

Let $\frac{7t+19}{(t+3)(t-5)} = \frac{A}{t+3} + \frac{B}{t-5}$

$\Rightarrow \quad 7t + 19 = A(t - 5) + B(t + 3)$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$\therefore \quad \frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{1}{4(t+3)} + \frac{27}{4(t-5)}$

$\Rightarrow \quad I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx = \int dx + \frac{1}{4}\int \frac{1}{(x^2+3)}dx$

$+ \frac{27}{4}\int \frac{1}{(x^2-5)}dx$

$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{4}\times\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$

$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$

28. Let $I = \int \frac{x}{(x^2+1)(x-1)}dx$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ ...(1)

$\Rightarrow \quad x = (Ax + B)(x - 1) + C(x^2 + 1)$ ...(2)

Comparing coefficients of x^2, x and constant terms, we get

$A + C = 0;\ B - A = 1;\ -B + C = 0$

Solving these, we get

$A = -\frac{1}{2}, C = \frac{1}{2}, B = \frac{1}{2}$

$\therefore$ From (1), we get

$\frac{x}{(x^2+1)(x-1)} = \frac{-\frac{1}{2}(x-1)}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x-1}$

$= -\frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x-1}$

$\therefore \quad I = -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x-1}$

$\Rightarrow \quad I = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C_1$

29. Let $I = \int e^{2x}\sin(3x+1)dx$

$= e^{2x}\int \sin(3x+1)dx - \int\left(\frac{d(e^{2x})}{dx}\cdot\int \sin(3x+1)dx\right)dx$

$= e^{2x}\frac{[-\cos(3x+1)]}{3} - \int 2e^{2x}\cdot\frac{[-\cos(3x+1)]}{3}dx$

$= \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{3}\int e^{2x}\cos(3x+1)dx$

$= \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{3}\Big[e^{2x}\int \cos(3x+1)dx$

$- \int\left(\frac{d}{dx}(e^{2x})\cdot\int \cos(3x+1)dx\right)dx\Big]$

$= \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{9}e^{2x}\sin(3x+1)$

$- \frac{4}{9}\int e^{2x}\sin(3x+1)dx$

$= \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{9}e^{2x}\sin(3x+1) - \frac{4}{9}I + C_1$

$\therefore \quad I + \frac{4}{9}I = \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{9}e^{2x}\sin(3x+1) + C_1$

$$\Rightarrow \quad \frac{13I}{9} = \frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{9}e^{2x}\sin(3x+1) + C_1$$

$$\Rightarrow \quad I = \frac{9}{13}\left[\frac{-e^{2x}\cos(3x+1)}{3} + \frac{2}{9}e^{2x}\sin(3x+1) + C_1\right]$$

$$= \frac{9}{13}e^{2x}\left[\frac{2\sin(3x+1) - 3e^{2x}\cos(3x+1)}{9}\right] + \frac{9}{13}C_1$$

$$= \frac{1}{13}e^{2x}[2\sin(3x+1) - 3\cos(3x+1)] + C$$

where $C = \frac{9}{13}C_1$

30. Let $I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$\Rightarrow \quad I = \int \frac{\cos\theta(\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)d\theta \quad \Rightarrow \quad I = -\int \theta\cos\theta d\theta$$

$$\Rightarrow \quad -I = \theta\int \cos\theta\, d\theta - \int\left(\frac{d}{d\theta}\theta\int\cos\theta d\theta\right)d\theta$$

$$\Rightarrow \quad -I = \theta\sin\theta - \int\sin\theta d\theta \quad \Rightarrow \quad -I = \theta\sin\theta + \cos\theta + C$$

$$\Rightarrow \quad I = -[\cos^{-1}x\sqrt{1-\cos^2\theta} + x] + C$$

$$\therefore \quad I = -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$$

31. Let $I = \int_0^{\pi/2} x^2 \sin x\, dx$

Integrating by parts, we get

$$I = \left[x^2(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)\,dx$$

$$= -\frac{\pi^2}{4}.0 + 0 + 2\int_0^{\pi/2} x\cos x\,dx = 2\int_0^{\pi/2} x\cos x\,dx$$

Again integrating by parts

$$I = 2\left[[x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx\right]$$

$$= 2\left\{\frac{\pi}{2}\cdot 1 - 0 - [-\cos x]_0^{\pi/2}\right\} \quad = 2\left[\frac{\pi}{2} + (0-1)\right] = \pi - 2$$

32. Let $I = \int_0^{\pi} e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx$

Put $\frac{\pi}{4} + x = t \Rightarrow x = t - \frac{\pi}{4} \Rightarrow dx = dt$

When $x = 0$, $t = \frac{\pi}{4}$ and when $x = \pi$, $t = \frac{5\pi}{4}$

$$\therefore \quad I = \int_{\pi/4}^{5\pi/4} e^{2\left(t-\frac{\pi}{4}\right)}\sin t\,dt = e^{-\pi/2}\int_{\pi/4}^{5\pi/4} e^{2t}\sin t\,dt$$

$$= e^{-\pi/2}\left[\left(\sin t\frac{e^{2t}}{2}\right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4}\cos t\frac{e^{2t}}{2}dt\right]$$

$$= e^{-\pi/2}\left[\frac{1}{2}\left(e^{5\pi/2}\sin\frac{5\pi}{4} - e^{\pi/2}\sin\frac{\pi}{4}\right)\right.$$
$$\left. -\left(\frac{e^{2t}}{4}\cos t\right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4}\frac{e^{2t}}{4}\sin t\,dt\right]$$

$$= e^{-\pi/2}\left[\frac{1}{2}\left(\frac{-1}{\sqrt{2}}e^{5\pi/2} - \frac{1}{\sqrt{2}}e^{\pi/2}\right)\right.$$
$$\left. -\frac{1}{4}\left(-\frac{1}{\sqrt{2}}e^{5\pi/2} - \frac{1}{\sqrt{2}}e^{\pi/2}\right)\right] - \frac{I}{4}$$

$$\Rightarrow \quad I + \frac{1}{4}I = -\frac{1}{2\sqrt{2}}[e^{2\pi}+1] + \frac{1}{4\sqrt{2}}[e^{2\pi}+1]$$

$$\Rightarrow \quad \frac{5}{4}I = \frac{(e^{2\pi}+1)}{2\sqrt{2}}\left[\frac{1}{2}-1\right] = -\frac{1}{4\sqrt{2}}[e^{2\pi}+1]$$

$$\Rightarrow \quad I = \frac{-1}{5\sqrt{2}}(1+e^{2\pi})$$

33. Let $I = \int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$

$$\Rightarrow \quad I = \int_0^{\pi}\frac{\pi - x}{1+\sin\alpha\sin(\pi-x)}dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx - \int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow \quad I = \int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx - I \quad \Rightarrow \quad 2I = \int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi}\frac{1}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

$$\left[\because \sin 2A = \frac{2\tan A}{1+\tan^2 A}\right]$$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

Also, when $x \to 0$, $t \to \tan 0 = 0$;

when $x \to \pi$, $t \to \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\infty}\frac{2dt}{t^2+2t\sin\alpha+1}$$

$$\Rightarrow \quad I=\pi\int_0^{\infty}\frac{1}{(t+\sin\alpha)^2+\cos^2\alpha}\,dt$$

$$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\infty-\tan^{-1}(\tan\alpha)\right] \Rightarrow I=\frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)$$

34. Let $I=\int_1^4(|x-1|+|x-2|+|x-4|)\,dx$

Also, let $f(x) = |x-1| + |x-2| + |x-4|$

We have three critical points $x = 1, 2, 4$.

$$f(x)=\begin{cases}(x-1)-(x-2)-(x-4), \text{ if } 1\le x<2\\(x-1)+(x-2)-(x-4), \text{ if } 2\le x<4\end{cases}$$

$$\therefore f(x)=\begin{cases}-x+5, \text{ if } 1\le x<2\\x+1, \text{ if } 2\le x<4\end{cases}$$

$$\therefore I=\int_1^4 f(x)dx=\int_1^2 f(x)dx+\int_2^4 f(x)dx$$

$$=\int_1^2(-x+5)\,dx+\int_2^4(x+1)\,dx=\left[-\frac{x^2}{2}+5x\right]_1^2+\left[\frac{x^2}{2}+x\right]_2^4$$

$$=\left(-\frac{4}{2}+10\right)-\left(-\frac{1}{2}+5\right)+\left(\frac{16}{2}+4\right)-\left(\frac{4}{2}+2\right)$$

$$=8-\frac{9}{2}+12-4=16-\frac{9}{2}=\frac{23}{2}$$

35. Let $I=\int_{\pi/6}^{\pi/3}\frac{dx}{(1+\sqrt{\tan x})}$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{dx}{\left(1+\sqrt{\frac{\sin x}{\cos x}}\right)}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad \text{...(1)}$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}\,dx$$

$$\left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx=\int_{\pi/6}^{\pi/3}dx$$

$$\Rightarrow \quad 2I=[x]_{\pi/6}^{\pi/3}=\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{6} \Rightarrow 2I=\frac{\pi}{6}$$

$$\Rightarrow \quad I=\frac{\pi}{12}$$

36. Let $I=\int\frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx=\int\frac{6x+7}{\sqrt{x^2-9x+20}}\,dx$

Let $6x+7=A\left[\frac{d}{dx}(x^2-9x+20)\right]+B$

$\therefore \quad 6x + 7 = A[2x - 9] + B$

Equating the coefficients of like terms from both sides, we get

$2A = 6$ and $-9A + B = 7$

$\Rightarrow \quad A = 3$ and

$-9(3) + B = 7 \quad \Rightarrow \quad B = 7 + 27 = 34$

$$\therefore \quad I=\int\frac{3(2x-9)}{\sqrt{x^2-9x+20}}\,dx+\int\frac{34}{\sqrt{x^2-9x+20}}\,dx$$

Put $x^2 - 9x + 20 = t$ in first integral

$$\therefore \quad I=\int\frac{3}{\sqrt{t}}\,dt+34\int\frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2+20-\frac{81}{4}}}$$

$$=3\int t^{-1/2}dt+34\int\frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}}$$

$$=3\frac{t^{1/2}}{1/2}+34\int\frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=6\sqrt{t}+34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|+C$$

37. Let $I=\int\frac{x^2+1}{(x-1)^2(x+3)}\,dx$

Let $\frac{x^2+1}{(x-1)^2(x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}$

$\Rightarrow \quad x^2+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \quad \text{...(1)}$

Put $x = 1$ in (1), we get $B=\frac{1}{2}$

Put $x = -3$ in (1), we get $C=\frac{5}{8}$

Put $x = 0$ in (1), we get $A=\frac{3}{8}$

$$\therefore \quad \frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$$

Integrating both sides, we get

$$I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\int\frac{dx}{(x-1)} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2}\cdot\frac{1}{(x-1)} + \frac{5}{8}\log|x+3| + C_1$$

38. Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx, x \in [0,1]$

We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\sqrt{x} = \frac{\pi}{2} - \cos^{-1}\sqrt{x}$$

$$\therefore \quad I = \int \frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\pi/2}\,dx = \int 1\cdot dx - \frac{4}{\pi}\int 1\cdot\cos^{-1}\sqrt{x}\,dx$$

$$= x - \frac{4}{\pi}\left[x\cdot\cos^{-1}\sqrt{x} - \int x\cdot\frac{-1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}\,dx\right] + C$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore I = x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int(1-\cos 2\theta)\,d\theta + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\theta - \sin\theta\cos\theta] + C$$

$$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}] + C$$

39. L.H.S. $= \int_0^{\pi/4}(\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right)dx = \int_0^{\pi/4}\frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4}\frac{(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}\,dx = \sqrt{2}\int_0^{\pi/4}\frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4}\frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

Let $\sin x - \cos x = t$, then $(\cos x + \sin x)\,dx = dt$

Also, $x = 0 \Rightarrow t = -1$ and $x = \pi/4 \Rightarrow t = 0$.

$$\therefore \quad \int_0^{\pi/4}(\sqrt{\tan x} + \sqrt{\cot x})\,dx = \sqrt{2}\int_{-1}^{0}\frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\left[\sin^{-1}t\right]_{-1}^{0} = \sqrt{2}\,[\sin^{-1}0 - \sin^{-1}(-1)]$$

$$= \sqrt{2}\cdot\sin^{-1}1 = \sqrt{2}\cdot\frac{\pi}{2} = \text{R.H.S.}$$

40. Let $I = \int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$ ...(1)

$$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)dx}{a^2\cos^2 x + b^2\sin^2 x}$$ ...(2)

Adding (1) and (2), we get

$$I = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Let $f(x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$

$$\Rightarrow \quad f(\pi - x) = \frac{1}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$$

$$\Rightarrow \quad f(\pi - x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x} = f(x)$$

$$\left[\text{using} \int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a - x) = f(x)\right]$$

$$\therefore \quad I = \frac{\pi}{2}\left(2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right)$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$.

Also when $x = 0 \Rightarrow t = \tan 0 = 0$.

And when $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty}\frac{dt}{a^2 + b^2t^2} \Rightarrow I = \frac{\pi}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$\Rightarrow \quad I = \frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \quad I = \frac{\pi}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{\pi^2}{2ab}$$